Краткие сообщения ОИЯИ №4[24]-87 *JINR Rapid Communications No.4[24]-87*
УДК 517.95.951

APPROXIMATE DIFFERENTIAL FADDEEV-TYPE EQUATIONS FOR SYSTEMS OF ONE LIGHT AND TWO HEAVY PARTICLES

V.V.Pupyshev

Three-particle integro-differential equations are explored. The double-sum-representation of nonlocal operators is obtained. It is shown, that nonlocal operators may be approximate by the sum of local one, if the system consists of one light and two heavy particles. After such approximation original equations are reduced to approximate partial differential one.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Приближенные дифференциальные уравнения фаддеевского типа для систем из одной легкой и двух тяжелых частиц

В.В.Пупышев

Исследуются интегродифференциальные уравнения для системы трех частиц. Для нелокальных операторов получено представление в виде двухкратных сумм. Показано, что для систем из одной легкой и двух тяжелых частиц такие операторы могут быть аппроксимированы суммой локальных операторов. Исходные уравнения после такой аппроксимации сводятся к приближенным дифференциальным уравнениям в частных производных.

Рабора выполнена в Лаборатории теоретической физики ОИЯИ.

Now the theory of three nonrelativistic interacting particles is intensively developed in the framework of integro-differential equations /1/. They are obtained /2/ from the system of equations

$$(H_0 - E + V_i)\Psi_i = -V_i \sum_{k \neq i} \Psi_k, \quad i = 1,2,3 \tag{1}$$

by decomposing of searched Faddeev components Ψ_i in a series

$$\Psi_i(\vec{x}_i, \vec{y}_i) = \sum_{\alpha=(\lambda,\ell),L} \Phi_i^{\alpha L}(x_i, y_i)(x_i y_i)^{-1} Y_\alpha^L(\hat{y}_i, \hat{x}_i) \tag{2}$$

over bispherical harmonics $Y_\alpha^L(\hat{y}, \hat{x})$. Here $\hat{a} = (\theta_a, \phi_a)$ are spherical angles of vector $\vec{a}$ in a fixed Cartezian coordinate system $S = \{\vec{e}_1, \vec{e}_2,$

$\vec{e}_3\}$. Different sets $(i \neq k)$ of relative Jacobi coordinates are connected with each other by the unitary transformation

$$\begin{pmatrix} \vec{x}_k \\ \vec{y}_k \end{pmatrix} = -\cos\gamma_{ki} \begin{pmatrix} 1 & -\epsilon_{ki}\,\mathrm{tg}\,\gamma_{ki} \\ \epsilon_{ki}\,\mathrm{tg}\,\gamma_{ki} & 1 \end{pmatrix} \begin{pmatrix} \vec{x}_i \\ \vec{y}_i \end{pmatrix}. \tag{3}$$

Kinematical angles $\gamma_{ki} \in [0, \pi/2]$ are determined only by the ratios of particle masses

$$\mathrm{tg}^2\gamma_{ki} = m_j M/m_i m_k = m_j/m_i + m_j/m_k + (m_j/m_i)(m_j/m_k), \tag{4}$$

and numbers ϵ_{ki} are such that $\epsilon_{ki} = -\epsilon_{ik} = 1$, $(ik) = (12), (31), (23)$. After substitution of components (2) into equations (1) and projection on the bispherical basis, the system (1) is reduced to a system of integro-differential equations for partial components Φ_i^{aL}. If the two-body potentials are central, then such a system is written in the following form /1, 2/

$$\{\Delta_i^a - E + V_i(x_i)\}\Phi_i^{aL}(x_i, y_i) = V_i(x_i) \sum_{k \neq i,\, a'} \langle x_i, y_i \,|\, \hat{h}^L_{aa'} \,|$$

$$|\,\Phi_k^{a'L}(x_k, y_k)\rangle, \quad i = 1,2,3, \quad a = (\lambda, \ell), \; a' = (\lambda', \ell'), \; \vec{\lambda} + \vec{\ell} = \tag{5}$$

$$= \vec{L} = \vec{\lambda}' + \vec{\ell}', \quad \Delta_i^a = \partial^2_{x_i} + \partial^2_{y_i} - \ell(\ell+1)/x_i^2 - \lambda(\lambda+1)/y_i^2 .$$

Matrix elements of $\hat{h}$ -operators in the right-hand side of equations (5) are determined as integrals

$$xy \int d\hat{x}\, d\hat{y}\, Y_a^{L*}(\hat{y}, \hat{x})\, Y_{a'}^{L}(\hat{y}', \hat{x}')\, \Phi_k^{a'L}(x', y')/(x'y'). \tag{6}$$

Here and further, when it is possible, indices i and k are omitted and, instead of the latter, the upper prime is used. Let us now obtain a representation more convenient for our investigation of system (5), and more compact than the known one /1/, for $\hat{h}$ -operators. We denote a plane going through three particles by symbol $\mathcal{P}$ and introduce the new coordinate system

$$S' = \{\vec{e}_1{}', \vec{e}_2{}', \vec{e}_3{}'\}, \qquad \vec{e}_1{}', \vec{e}_2{}' \in \mathcal{P}, \quad \vec{e}_3{}' \uparrow\uparrow \vec{x}.$$

The original system S is obtained by rotation of the new system S'. This rotation is determined by Euler angles $\omega = \{\omega_1, -\theta_x, \mp\phi_x\}$ where ω_1 is the angle of the first rotation around $\vec{e}_3{}'$ axis, upon which vector $\vec{e}_2{}'$

gets coincident with the normal to the $\mathcal{P}$ plane. The arguments marked by primes in formulae (6) are functions (3) of independent variables $\rho = (x^2 + y^2)^{1/2}$, $\phi = \mathrm{arctg}(y/x)$, $u = \cos(\hat{\vec{x}\vec{y}})$. This allows us to express bispherical harmonics in integrals (6) by a linear combination of $D^L(\omega)$-Wigner functions /4/ and bispherical harmonics written in the new system S′ to replace the variables $d\hat{x}d\hat{y} = d\omega\, du$ and to integrate over Euler angles. As a final result, matrix element (6) is written as the one-dimensional integral

$$<x,y|\hat{h}^L_{aa'}|\Phi^{a'L}(x',y')> = \int_{-1}^{1} du\, h^L_{aa'}(\phi,u,\epsilon\gamma)\Phi^{a'L}(x',y'), \qquad (7)$$

where

$$h^L_{aa'}(\phi,u,\epsilon\gamma) = J(\phi,\phi')\cdot\frac{[\lambda][\ell][\lambda'][\ell']}{2L+1}\cdot$$

$$\cdot\sum_{m,N} C^{LN}_{\lambda N\ell 0} C^{LN}_{\lambda' m\ell' n}\left\{\frac{(\lambda-N)!(\lambda'-m)!(\ell'-N+m)!}{(\lambda+N)!(\lambda'+m)!(\ell'+N-m)!}\right\}^{1/2}\cdot$$

$$\cdot P^N_{\lambda'}(u)P^m_{\lambda'}(u_{xy'})P^n_{\ell'}(u_{xx'}), \qquad (8)$$

here

$[a] = (2a+1)^{1/2}$, $J(\phi,\phi') = \sin 2\phi/2\sin 2\phi'$.

and all arguments ϕ', $u_{ab} \equiv \cos(\hat{\vec{a}\vec{b}})$ are functions (3) of variables ϕ,u and parameter $\epsilon\gamma$. The representation of $\hat{h}$ operators thus obtained is more convenient and compact than that usually used /1/. The latter is written as a five-index-sum containing 6j and 9j -symbols.

System (5) in the polar coordinates is written in the following form

$$\{\Delta^a_i - E + V_i(x_i)\}\Phi^{aL}_i(\rho,\phi_i) = V_i(x_i)\sum_{k\neq i,a'}<\phi_i|\hat{h}^L_{aa'}|\Phi^{a'L}_k(\rho,\phi_k)>, \qquad (9)$$

where $\Delta^a_i = \partial^2_\rho + \rho^{-1}\partial_\rho + \rho^{-2}(\partial^2_{\phi_i} - \frac{\ell(\ell+1)}{\cos^2\phi_i} - \frac{\lambda(\lambda+1)}{\sin^2\phi_i})$, and $x_i = \rho\cos\phi_i$. Matrix elements

$$<\phi|\hat{h}^L_{aa'}|\Phi^{a'L}_k(\rho,\phi')> = \int_{c(\phi,\gamma)}^{d(\phi,\gamma)} d\phi' h^L_{aa'}(\phi,\phi',\epsilon\gamma)\Phi^{a'L}_k(\rho,\phi'), \qquad (10)$$

where

$$h^{L}_{aa'}(\phi,\phi',\epsilon\gamma) = (2\,\mathrm{cosec}\,2\gamma/J)\,h^{L}_{aa'}(\phi,u(\phi,\phi',\epsilon\gamma),\epsilon\gamma), \tag{11}$$

and the integral limits are equal to

$$c(\phi,\gamma) = |\phi-\gamma|, \quad d(\phi,\gamma) = \min\{\phi+\gamma, \pi-\phi-\gamma\}, \tag{12}$$

are obtained [3] by replacement $u \to \phi'$ in corresponding integrals (7). The presence of nonlocal operators (10-12) in the system (9) essentially complicates both its numerical solution and the investigation of analytical properties of unknown partial components. Consequently, the study of these operators is an actual problem.

Let us show that operators $\hat{h}$ may be approximated by a sum of local operators if the particle masses and functions are subjected to well-defined constraints. First, we explore the mapping (10-12) in two limits $\gamma \to 0$, and $\gamma \to \pi/2$. The kernels (8), (11) of operators $\hat{h}$ are regular functions of parameter $\gamma \in [0,\pi/2]$ according to definition (6). Using equalities (3), (6) and known parity properties of bispherical harmonics [4], we obtain from formulae (6-8) limit forms of equalities (10)

$$\langle\phi|\hat{h}^{L}_{aa'}|\Phi^{a'L}_{k}(\rho,\phi')\rangle = g^{aa'L}_{o}(\phi,\epsilon\gamma)\Phi^{a'L}_{k}(\rho,\xi),$$

where

$$\xi=\phi, \quad g^{aa'L}_{o}(\phi,\epsilon\gamma) = (-1)^{\lambda+\ell}\,\delta_{aa'}, \qquad \text{if } \gamma=0,$$

$$\xi=\pi/2-\phi, \quad g^{aa'L}_{o}(\phi,\epsilon\gamma) = (-\epsilon)^{\lambda+\ell}\,\delta_{\lambda\ell'}\,\delta_{\lambda'\ell}, \quad \text{if } \gamma=\pi/2.$$

Consequently, if $\gamma=0$, or $\gamma=\pi/2$, then operators h are local. If γ tends to zero, then both limits (12) of integral (10) tend uniformly to ϕ, and the length of the integral interval is equal to $O(\gamma)$. If a partial angular derivative of order m (we denote it by ${}^{(m)}\Phi^{a'L}_{k}(\rho,\phi')$) is continuous everywhere, then, decomposing the partial component in the integrand of (10), in a Taylor series at the centre ϕ, we obtain the equality

$$\langle\phi|\hat{h}^{L}_{aa'}|\Phi^{a'L}_{k}(\rho,\phi')\rangle = \sum_{n=o}^{m} g^{aa'L}_{n}(\phi,\epsilon\gamma)\,{}^{(m)}\Phi^{a'L}_{k}(\rho,\phi) + r_{m}. \tag{13}$$

The functions $g^{aa'L}_{n}$ represent matrix elements $\langle\phi|\hat{h}^{L}_{aa'}|(\phi'-\phi)^{n}\rangle/n!$, which can be easily calculated by equalities (8), (10-12). From the latter equalities it follows that the $g_n^{aa'L}$ function is of an order of $O(\gamma^n/(n+1)!)$, and the remaining term $r_m \sim O(\gamma^m)$.

As an example, we cite functions $g_n^{aa'L}$ in the case $a = a' = (0,0)$, $\epsilon = 1$, $L = 0$:

$$g_n^{aao}(\phi,\gamma) = \gamma^{n+1}\operatorname{cosec}2\gamma/(n+1)! \begin{cases} 1-(1-2\phi/\gamma)^{n+1}, & \phi \in [0,\gamma] \\ 1+(-1)^n, & \phi \in [\gamma,\pi/4]. \end{cases}$$

These functions have parity $(-1)^n$ with respect to point $\phi = \pi/4$. Odd functions are not equal to zero only on subintervals $(0,\gamma)$ and $(\pi/2-\gamma, \pi/2)$. The lengths of the above intervals approach zero as $O(\gamma)$ when $\gamma \to 0$. If the particle masses are such that $\beta^2 = m_1/m_2 \ll 1$, $m_2 \sim m_3$, then kinematical angles are equal to $\gamma_{23} = 0 + O(\beta)$, γ_{12}, $\gamma_{13} = \pi/2 + O(\beta)$ according to definition (4). Therefore, approximation (13) of nonlocal operators $\hat{h}$ by a sum of local operators, i.e., by operators ∂_ϕ^n with weght-multipliers $g_n^{aa'L} \sim O(\beta^n)$ is kinematically feasible when a three-particle system consists of one light and two heavy pat- ticles. The class of such systems is sufficiently wide; it includes meso- molecular $(dd\mu)$, atomic $(e^\pm pp)$ systems, systems consisting of one nucleon and two heavy nuclei, and so on. Equations (9) are reduced to partial differential equations by approximation (13). It is necessary to point out, that approximation (13) does not change a free three- particle Hamiltonian in contrast to the well-known Born-Oppenheimer method /5/. Moreover, approximation (13) is a geometrical approximation only over the angle variables. A small parameter $\beta = (m_1/m_2)^{1/2}$ of approximation (13) has a kinematical nature and does not depend both an the total energy of a three-patricle system and on the shape of two-body interactions. The existence and uniqueness of the solution of equations (9) have been proved /1/ under the assumption $\forall\, \Phi_i^{aL} \in C^2$, therefore the rank m of approximation (13) obeys in equality $m \geq 2$.

The determination of maximal possible rank m of approximation (13) depending on the shape of potentials requires further investigations. The most interesting application of approximate equations, in my opinion, is an investigation of local analytic properties (first of all, the asymptotical behaviour in a vicinity of point $\rho = 0$) of partial components. Such investigations seem impossible in the framework of original exact integro-differential equations (9).

In conclusion, we briefly summarize main results of the present work. On the basis of the obtained representation (8) for kernels of operators $\hat{h}$ it is shown that these operators are reduced to the local ones in the limits $\gamma \to 0, \gamma \to \pi/2$. Such situation is realised for systems consisting of one light and two heavy particles. For such systems integro-differential equations may be replaced by approximate partial differential equations.

References

1. Меркурьев С.П., Фадеев Л.Д. Квантовая теория рассеяния для систем нескольких частиц. М.: Наука, 1985.
2. Noyes H.P. — In: Tree-Body Problem, North-Holland, Amsterdam, 1970, p.2.
3. Пупышев В.В. — ЯФ, 1986, 43, с.1318.
4. Варшалович Д.А., Москалев А.Н., Херсонский В.К. Квантовая теория углового момента. Л.: Наука, 1975.
5. Born M., Oppenheimer J. — Ann. Phys., 1927, 84, p.457.

Received on April 27, 1987.